# Some Properties of Roller Coaster Permutations

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**Abstract**

A Roller Coaster permutation is a permutation, along with all of its subsequences, that changes from increasing to decreasing (and vice versa) a maximum number of times. We offer a few conjectures (enumerative as well as structural) along with data describing some surprising properties of these permutations.

## 1 Introduction

A permutation $\pi$ of $[n] = \{1, 2, \ldots, n\}$ is a sequence $(\pi_1, \pi_2, \ldots, \pi_n)$. We omit commas and parenthesis when doing so produces no ambiguity. Let $S_n$ denote the set of all permutations of $[n]$. Now, we define the following (a sequence of contiguous numbers means at least 'two numbers'), where for a word $x$, $|x|$ denotes the number of letters in $x$:

$$\begin{aligned}
i(\pi) &= \#\text{ increasing sequences of contiguous numbers in } \pi,\\
d(\pi) &= \#\text{ decreasing sequences of contiguous numbers in } \pi,\\
id(\pi) &= i(\pi) + d(\pi),\\
X(\pi) &= \{\tau : \tau \text{ is a subsequence of } \pi \text{ such that } |\tau| \geqslant 3\},\\
t(\pi) &= \sum_{\tau \in X(\pi)} id(\tau).
\end{aligned}$$

For example,

$$\begin{aligned} t(2143) &= id(2143) + id(214) + id(213) + id(243) + id(143) \\ &= 3 + 2 + 2 + 2 + 2 = 11 \text{ and} \\ t(1234) &= id(1234) + id(123) + id(124) + id(134) + id(234) \\ &= 1 + 1 + 1 + 1 + 1 = 5. \end{aligned}$$

A permutation $\pi \in S_n$ is called a *Roller Coaster permutation* if $t(\pi) = \max_{\tau \in S_n} t(\tau)$. In this paper, we explore properties that enumerate and characterize Roller Coaster permutations.

## 2 Results on Roller Coaster permutations

Let $RC(n)$ be the set of Roller Coaster permutations in $S_n$. If $\pi \in RC(n)$, then the reverse of $\pi$ is also in $RC(n)$. Similarly, $\pi^* = (n+1-\pi_1, n+1-\pi_2, \ldots, n+1-\pi_n)$, the 'mod $n+1$' complement of $\pi$, is also in $RC(n)$. Let $t_{max}(n)$ be defined as $\max_{\pi \in S_n} t(\pi)$. We have the following experimental results for $n = 3, 4, \ldots, 24$ (conjectured values and lower bounds are in italics):

$$t_{max}: \quad [2, 11, 37, 106, 270, 653, 1523, 3480, 7768, 17123, 37405, \mathit{81350}, \mathit{174954}, \mathit{374409}, \mathit{798471}, \mathit{1700036}, \mathit{3596124}, \mathit{7588303}, \mathit{15970785}, \mathit{33596706}, \mathit{70310126}, \mathit{146867861}].$$

Here, we provide $RC(n)$ for $3 \leqslant n \leqslant 9$ (and $RC(n)$ for $10 \leqslant n \leqslant 13$ are provided in Appendix A):

RC(3) = {132, 213, 231, 312},
RC(4) = {2143, 2413, 3142, 3412},
RC(5) = {24153, 25143, 31524, 32514, 34152, 35142, 41523, 42513},
RC(6) = {326154, 351624, 426153, 451623},
RC(7) = {3517264, 3527164, 3617254, 3627154, 4261735, 4271635, 4361725, 4371625, 4517263, 4527163, 4617253, 4627153, 5261734, 5271634, 5361724, 5371624},
RC(8) = {43718265, 46281735, 53718264, 56281734},
RC(9) = {471639285, 471936285, 472639185, 472936185, 481639275, 481936275, 482639175, 482936175, 528174936, 528471936, 529174836, 529471836, 538174926, 538471926, 539174826, 539471826, 571639284, 571936284, 572639184, 572936184, 581639274, 581936274, 582639174, 582936174, 628174935, 628471935, 629174835, 629471835, 638174925, 638471925, 639174825, 639471825}.

A permutation $\pi$ be called *alternating* and *reverse-alternating* if $\pi_1 < \pi_2 > \cdots$ and $\pi_1 > \pi_2 < \cdots$, respectively. Clearly, for $n \geqslant 3$ and $\pi \in S_n$, we have $id(\pi) \leqslant n-1$ and

$$t_{max}(n) \leqslant \sum_{k=3}^{n} \binom{n}{k}(k-1).$$

Based on the data above, one can make the following conjectures:

**Conjecture 2.1.** If $\pi \in RC(n)$, then $\pi$ is alternating or reverse-alternating.

**Conjecture 2.2.** There exists $\pi \in RC(n)$, such that $\pi_1 = \lfloor n/2 \rfloor$ and $\pi_n = \lfloor n/2 \rfloor + 1$.

Let $f_t(\pi)$ be obtained from $\pi$ by swapping $\pi_i$ and $\pi_{n-i+1}$ for $t+1 \leqslant i \leqslant \lfloor n/2 \rfloor$. For example, $f_2(43718265) = 43281765$. Note that $f_0(\pi)$ is the plain reverse of $\pi$.

**Lemma 2.1.** If $\pi \in RC(n)$ where $n = 2k$ and Conjecture 2.2 is true, then $f_1(\pi) \in RC(n)$.

*Proof.* Consider $\pi = (k, \pi_2, \pi_3, \ldots, \pi_{n-1}, k+1) \in RC(n)$ where $k = \lfloor n/2 \rfloor$. Suppose $\tau = (s_1, s_2, \ldots, s_i)$ is a subsequence of $\pi$ where $0 \leqslant i \leqslant n-2$. Now we have the following four cases:

$(i)$ If $\tau$ involves neither $k$ nor $k+1$, then

$$id(s_1, s_2, \ldots, s_i) = id(s_i, s_{i-1}, \ldots, s_1).$$

$(ii)$ If $\tau$ involves only $k$, then

$$id(k, s_1, s_2, \ldots, s_i) = id(s_i, s_{i-1}, \ldots, s_1, k) = id(s_i, s_{i-1}, \ldots, s_1, k+1),$$

where the last equality results from $k$ and $k+1$ being indistinguishable in the context. Since the last term in both $\pi$ and $f_1(\pi)$ is $k+1$, subsequences involving $k$ in $\pi$ has as many runs as subsequences in $f_1(\pi)$ invloving $k+1$.

$(iii)$ If $\tau$ invloves only $k+1$, then the argument is similar as case $(ii)$.

$(iv)$ If $\tau$ involves both $k$ and $k+1$, then since $k$ and $k+1$ are indistinguishable, we have $id(k, \pi_2, \ldots, \pi_{n-1}, k+1) = id(k, \pi_{n-1}, \ldots, \pi_2, k+1)$.

Therefore, $t(f_1(\pi)) = t_{max}(n)$ and hence $f_1(\pi) \in RC(n)$. □

**Conjecture 2.3.** For $n \geqslant 3$,

$$|RC(n)| = \begin{cases} 4 & \text{if } n = 2k, \\ 2^j \text{ where } j \leqslant k+1 & \text{if } n = 2k+1. \end{cases}$$

If $n = 2k$, $k \geqslant 2$, and $\pi \in RC(n)$, then using Lemma 2.1 and the fact that reverse of a Roller Coaster permutation is also Roller Coaster, we get the following permutations in $RC(n)$:

$$\pi, f_0(\pi), f_1(\pi), f_0(f_1(\pi)),$$

that is, $|RC(n)| \geqslant 4$ if $n = 2k$ and $k \geqslant 2$.

For example, given $326154 \in RC(6)$, we obtain
$f_0(326154) = 451623$,
$f_1(326154) = 351624$,
$f_0(f_1(326154)) = f_0(351624) = 426153$.

**Conjecture 2.4** (The Odd Sum conjecture)**.** If $\pi \in RC(n)$ and $n = 2k+1$, then $\pi_j + \pi_{n-j+1}$ is odd for $1 \leqslant j \leqslant k$. If $\pi \in RC(n)$ and $n = 2k$, then $\pi_j + \pi_{n-j+1} = 2k + 1$ for all $1 \leqslant j \leqslant k$.

**Conjecture 2.5.** If $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in RC(n)$, then $RC(n)$ can be completely determined from $\pi$.

Let $g_L(\pi)$ be obtained from $\pi$ by swapping $\pi_i$ and $\pi_{n-i+1}$ for each $i \in L$ where $1 \leqslant i \leqslant \lfloor n/2 \rfloor$. For example, $g_{\{2,3,4\}}(471639285) = 482936175$.

If $n = 2k + 1$, $\pi \in RC(n)$, and Conjecture 2.3 is true, then we believe $RC(n)$ consists of $2^j$ ($j \leqslant k + 1$) permutations from the $2^{k+1}$ permutations obtained by taking $L$ as each element in the set $\mathcal{P}(\{1, 2, \ldots, k\})$ (the power set of $\{1, 2, \ldots, k\}$), and the 'mod $n + 1$' complement of each of these $2^k$ permutations. This algorithm works for $n = 3, 5, 7, 9$, and 11 where all Roller Coaster permutations are enumerated, that is,

$$|RC(2k + 1)| = 2^{k+1} \text{ for } k = 1, 2, 3, 4, 5.$$

For example, given $\pi = 3517264 \in RC(7)$, we obtain

1. $g_\emptyset(\pi) = 3517264$; $g_\emptyset(\pi)^* = 5371624$,
2. $g_{\{1\}}(\pi) = 4517263$; $g_{\{1\}}(\pi)^* = 4371625$,
3. $g_{\{2\}}(\pi) = 3617254$; $g_{\{2\}}(\pi)^* = 5271634$,
4. $g_{\{3\}}(\pi) = 3527164$; $g_{\{3\}}(\pi)^* = 5361724$,
5. $g_{\{1,2\}}(\pi) = 4617253$; $g_{\{1,2\}}(\pi)^* = 4271635$,
6. $g_{\{1,3\}}(\pi) = 4527163$; $g_{\{1,3\}}(\pi)^* = 4361725$,
7. $g_{\{2,3\}}(\pi) = 3627154$; $g_{\{2,3\}}(\pi)^* = 5261734$, and
8. $g_{\{1,2,3\}}(\pi) = 4627153$; $g_{\{1,2,3\}}(\pi)^* = 4261735$.

Given $\pi \in S_{2k+1}$, define:

$$\begin{aligned} g(\pi) &= \{\tau, \tau^* : g_L(\pi) = \tau \text{ for some } L \in \mathcal{P}(\{1, 2, \dots, k\})\} \\ RC(\pi) &= \left\{\tau : \tau \in g(\pi) \text{ and } t(\tau) = \max_{\sigma \in g(\pi)} t(\sigma)\right\}. \end{aligned}$$

Here $\tau^*$ is the 'mod $2k+2$' complement of $\tau$. Note that, $RC(n)$ is different from $RC(\pi)$ as the latter is the set of all permutations in $g(\pi)$ (instead of $S_n$) with maximal $t$.

## 2.1 Fast computation of lower bounds of $t_{max}(n)$

In this section, we propose a very fast algorithm to compute a permutation $\pi \in S_n$ such that $t(\pi)$ gives a lower bound for $t_{max}(n)$. The following heuristics act as a guide for the algorithm proposed in this section:

- $\pi$ satisfies Conjecture 2.1,
- $\pi$ satisfies Conjecture 2.2, and
- $\pi$ satisfies Conjecture 2.4.

### 2.1.1 $n = 2k$:

Lower bound of $t_{max}(2k)$ can be computed from a $\tau \in S_{k-1}$ by obtaining a permutation $\pi \in S_{2k}$ as follows:

$$\pi_i = \begin{cases} k & \text{if } i = 1, \\ k+1 & \text{if } i = 2k, \\ \tau_j & \text{if } i = 2j+1 \text{ for } 1 \leqslant j \leqslant k-1, \\ n+1-\tau_{k-j} & \text{if } i = 2j \text{ for } 1 \leqslant j \leqslant k-1. \end{cases}$$

Here $\tau_{k-j}$ represents $\pi_{n-i+1}$ when $i = 2j+1$ for $1 \leqslant j \leqslant k-1$.

1. For $k = 7$, $\tau = 351624 \in RC(6)$ gives

   $$\pi = (7, 11, 3, 13, 5, 9, 1, 14, 6, 10, 2, 12, 4, 8) \in S_{14}$$

   with $t_{max}(14) \geqslant 81350$.

   We observe that

   $$t(f_0(\pi)) = t(f_1(\pi)) = t(f_0(f_1(\pi))) = 81350.$$

2. For $k = 8$, $\tau = 4261735 \in RC(7)$ gives

$$\pi = (8, 12, 4, 14, 2, 10, 6, 16, 1, 11, 7, 15, 3, 13, 5, 9) \in S_{16}$$

with $t_{max}(16) \geqslant 374409$.
We observe that

$$t(f_0(\pi)) = t(f_1(\pi)) = t(f_0(f_1(\pi))) = 374409.$$

3. For $k = 9$, $\tau = 46281735 \in RC(8)$ gives

$$\pi = (9, 14, 4, 16, 6, 12, 2, 18, 8, 11, 1, 17, 7, 13, 3, 15, 5, 10) \in S_{18}$$

with $t_{max}(18) \geqslant 1699748$.
We observe that

$$t(f_0(\pi)) = t(f_1(\pi)) = t(f_0(f_1(\pi))) = 1699748.$$

See Section 2.2 for slight improvement in this bound.

4. For $k = 10$, $\tau = 528471936 \in RC(9)$ gives

$$\pi = (10, 15, 5, 18, 2, 12, 8, 20, 4, 14, 7, 17, 1, 13, 9, 19, 3, 16, 6, 11) \in S_{20}$$

with $t_{max}(20) \geqslant 7588303$.
We observe that

$$t(f_0(\pi)) = t(f_1(\pi)) = t(f_0(f_1(\pi))) = 7588303.$$

5. For $k = 11$, $\tau = (5, 8, 2, 10, 4, 7, 1, 9, 3, 6) \in RC(10)$ gives

$$\pi = (11, 17, 5, 20, 8, 14, 2, 22, 10, 16, 4, 19, 7, 13, 1, 21, 9, 15, 3, 18, 6, 12)$$

in $S_{22}$ with $t_{max}(22) \geqslant 33596706$.
We observe that

$$t(f_0(\pi)) = t(f_1(\pi)) = t(f_0(f_1(\pi))) = 33596706.$$

6. For $k = 12$, $\tau = (6, 3, 9, 1, 11, 5, 8, 2, 10, 4, 7) \in RC(11)$ gives the following $\pi \in S_{24}$

$$(12, 18, 6, 21, 3, 15, 9, 23, 1, 17, 11, 20, 5, 14, 8, 24, 2, 16, 10, 22, 4, 19, 7, 13)$$

with $t_{max}(24) \geqslant 146867861$.
We observe that

$$t(f_0(\pi)) = t(f_1(\pi)) = t(f_0(f_1(\pi))) = 146867861.$$

### 2.1.2 $n = 2k+1$:

Lower bound of $t_{max}(2k+1)$ can be computed from a $\tau \in S_k$ and $\rho \in S_{k-1}$ by obtaining a permutation $\pi \in S_{2k+1}$ as follows:

$$\pi_i = \begin{cases} k & \text{if } i = 1, \\ k+1 & \text{if } i = 2k+1, \\ \tau_j + k + 1 & \text{if } i = 2j \text{ for } 1 \leqslant j \leqslant k, \\ \rho_j & \text{if } i = 2j+1 \text{ for } 1 \leqslant j \leqslant k-1. \end{cases}$$

1. For $k = 7$, $\tau = 3517264 \in RC(7)$ and $\rho = 326154 \in RC(6)$ give

$$\pi = (7, 11, 3, 13, 2, 9, 6, 15, 1, 10, 5, 14, 4, 12, 8) \in S_{15}$$

with $t_{max}(15) \geqslant 174954$.

We observe that

$$t(\pi) = \max_{\sigma \in g(\pi)} t(\sigma) = 174954 \text{ and } |RC(\pi)| = 128 = 2^7.$$

2. For $k = 8$, $\tau = 43718265 \in RC(8)$ and $\rho = 3517264 \in RC(7)$ give

$$\pi = (8, 13, 3, 12, 5, 16, 1, 10, 7, 17, 2, 11, 6, 15, 4, 14, 9) \in S_{17}$$

with $t_{max}(17) \geqslant 798471$.

We observe that

$$t(\pi) = \max_{\sigma \in g(\pi)} t(\sigma) = 798471 \text{ and } |RC(\pi)| = 128 = 2^7.$$

3. For $k = 9$, $\tau = 471639285 \in RC(9)$ and $\rho = 43718265 \in RC(8)$ give

$$\pi = (9, 14, 4, 17, 3, 11, 7, 16, 1, 13, 8, 19, 2, 12, 6, 18, 5, 15, 10) \in S_{19}$$

with $t_{max}(19) \geqslant 3596124$.

We observe that

$$t(\pi) = \max_{\sigma \in g(\pi)} t(\sigma) = 3596124 \text{ and } |RC(\pi)| = 256 = 2^8.$$

4. For $k = 10$, $\tau = (5, 3, 9, 1, 7, 4, 10, 2, 8, 6) \in RC(10)$ and $\rho = 471639285 \in RC(9)$ give

$$\pi = (10, 16, 4, 14, 7, 20, 1, 12, 6, 18, 3, 15, 9, 21, 2, 13, 8, 19, 5, 17, 11) \in S_{21}$$

with $t_{max}(21) \geqslant 15970785$. We observe that

$$t(\pi) = \max_{\sigma \in g(\pi)} t(\sigma) = 15970785 \text{ and } |RC(\pi)| = 128 = 2^7.$$

5. For $k = 11$, $\tau = (5, 8, 2, 10, 1, 7, 4, 11, 3, 9, 6) \in RC(11)$ and $\rho = (5, 3, 9, 1, 7, 4, 10, 2, 8, 6) \in RC(10)$ give the following $\pi \in S_{23}$

$$(11, 17, 5, 20, 3, 14, 9, 22, 1, 13, 7, 19, 4, 16, 10, 23, 2, 15, 8, 21, 6, 18, 12)$$

with $t_{max}(23) \geqslant 70310126$. We observe that

$$t(\pi) = \max_{\sigma \in g(\pi)} t(\sigma) = 70310126 \text{ and } |RC(\pi)| = 256 = 2^8.$$

### 2.2 Choosing $\tau$ and $\rho$ optimally

The above algorithm produces a lower bound of $t_{max}(n)$ using a suitable choice of $\tau$ and $\rho$. In the case of $t_{max}(18)$, David Callan shows that taking $\tau = 47261835 \in S_8$ (note that $\tau \notin RC(8)$ as $t(\tau) = 651 < t_{max}(8)$) gives

$$\pi = (9, 14, 4, 16, 7, 11, 2, 18, 6, 13, 1, 17, 8, 12, 3, 15, 5, 10) \in S_{18}$$

with $t_{max}(18) \geqslant 1700036$, which is slightly better than the bound that can be obtained by taking a $\tau \in RC(8)$. So it is not necessarily optimal to choose an optimal $\tau \in RC(k-1)$. Again, if $n = 2k+1$, then we may let $\tau$ be the lexicographically least element of $RC(k)$ and let $\rho$ be the lexicographically least element in $RC(k-1)$. It remains open how to choose $\tau$ and $\rho$ optimally.

## 3 $t_{max}$ considering only subsequences of specific length

Here we consider a variant of $t_{max}$ defined as follows:

$$\begin{aligned}
X(\ell, \pi) &= \{\tau : \tau \text{ is a subsequence of } \pi \text{ such that } |\tau| = \ell\}, \\
t(\ell, \pi) &= \sum_{\tau \in X(\ell, \pi)} id(\tau), \\
t_{max}(\ell, n) &= \max_{\pi \in S_n} t(\ell, \pi), \\
RC(\ell, n) &= \{\pi \in S_n : t(\ell, \pi) = t_{max}(\ell, n)\}.
\end{aligned}$$

We have some experimental data on $t_{max}(k, n)$ based on which we have the following few conjectures:

| $\ell/n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 8 | 19 | 38 | 65 | 104 | 154 | 220 |
| 4 | | 3 | 14 | 41 | 93 | 184 | 328 | 541 |
| 5 | | | 4 | 22 | 75 | 194 | 430 | 852 |
| 6 | | | | 5 | 32 | 124 | 363 | 894 |
| 7 | | | | | 6 | 44 | 191 | 622 |
| 8 | | | | | | 7 | 58 | 279 |
| 9 | | | | | | | 8 | 74 |
| 10 | | | | | | | | 9 |

**Conjecture 3.1.** The lexicographically smallest permutation in $RC(n-1, n)$ is given by

$$\pi_i = \begin{cases} 2j-1 & \text{if } i = 2j, 1 \leqslant j \leqslant k, \\ 2j & \text{if } i = 2j-1, 1 \leqslant j \leqslant k-1, \end{cases}$$

with $(\pi_{n-1}, \pi_n) = (2k, 2k-1)$ if $n = 2k$, and $(\pi_{n-2}, \pi_{n-1}, \pi_n) = (2k+1, 2k-1, 2k)$ if $n = 2k+1$. For example, 21436587 and 214365978 are the lexicographically smallest permutations in $RC(7, 8)$ and $RC(8, 9)$, respectively.

**Claim 3.1.** Assuming Conjecture 3.1 is true, we have for $n \geqslant 4$,

$$t_{max}(n-1, n) = (n-1)(n-2) + 2.$$

*Proof.* Take the lexicographically smallest permutation $\pi \in RC(n-1, n)$. For both the parities of $n$, $\pi$ is reverse-alternating. Suppose $n = 2k$. There are $\binom{n}{n-1} = n$ subsequences of $\pi$ each of length $n-1$, contribute to $t_{max}(n-1, n)$ as follows:

$$\begin{aligned} t_{max}(n-1, n) &= \sum_{\substack{\tau=\pi\setminus\{a\}, \\ a\in\{2,1,2k,2k-1\}}} id(\tau) + \sum_{\substack{\tau=\pi\setminus\{a\}, \\ a\notin\{2,1,2k,2k-1\}}} id(\tau), \\ &= 4(n-2) + (n-4)(n-3) = (n-1)(n-2) + 2. \end{aligned}$$

If $n = 2k+1$, then the argument is similar as above. □

**Fact 3.1** (Myers [4])**.** The lexicographically smallest permutation in $RC(3, n)$ is

1. $(k, k-1, \ldots, 1, 2k, 2k-1, \ldots, k+1)$ if $n = 2k$, and
2. $(k, k-1, \ldots, 1, 2k+1, 2k, 2k-1, \ldots, k+1)$ if $n = 2k+1$.

**Lemma 3.1.** For $n \geqslant 3$,

$$t_{max}(3, n) = \begin{cases} k(k-1)(7k-2)/3 & \text{if } n = 2k, \\ k(14k^2 + 3k - 5)/6 & \text{if } n = 2k+1 \end{cases}$$

*Proof.* If $n = 2k$, take an optimal permutation $\pi = (k, k-1, \ldots, 1, 2k, 2k-1, \ldots, k+1) \in RC(3, 2k)$. Considering length-three subsequences $\tau$ of $\pi$, we have,

$$\begin{aligned} t_{max}(3, 2k) &= t(3, \pi) \\ &= \sum_{\substack{\tau=abc,\\ a>b>c}} id(\tau) + \sum_{\substack{\tau=abc,\\ a>b<c}} id(\tau) + \sum_{\substack{\tau=abc,\\ a<b>c}} id(\tau) \\ &= 2\binom{k}{3} \cdot 1 + \binom{k}{2} k \cdot 2 + k\binom{k}{2} \cdot 2 \\ &= k(k-1)(7k-2)/3. \end{aligned}$$

If $n = 2k+1$, take an optimal permutation $\pi = (k, k-1, \ldots, 1, 2k+1, 2k, 2k-1, \ldots, k+1) \in RC(3, 2k+1)$. Considering length-three subsequences $\tau$ of $\pi$, we have,

$$\begin{aligned} t_{max}(3, 2k+1) &= t(3, \pi) \\ &= \sum_{\substack{\tau=abc,\\ a>b>c}} id(\tau) + \sum_{\substack{\tau=abc,\\ a>b<c}} id(\tau) + \sum_{\substack{\tau=abc,\\ a<b>c}} id(\tau) \\ &= \left[\binom{k}{3} + \binom{k+1}{3}\right] \cdot 1 + \binom{k}{2}(k+1) \cdot 2 + k\binom{k+1}{2} \cdot 2 \\ &= k(14k^2 + 3k - 5)/6. \end{aligned}$$

□

**Conjecture 3.2** (Myers [4])**.** For $j \geqslant 3$, $t_{max}(j+1, n)$ is given by the following permutation in $S_n$

$$\left\lfloor \frac{n}{j} \right\rfloor, \left\lfloor \frac{n}{j} \right\rfloor - 1, \ldots, 1, \left\lfloor \frac{2n}{j} \right\rfloor, \left\lfloor \frac{2n}{j} \right\rfloor - 1, \left\lfloor \frac{n}{j} \right\rfloor + 1, \ldots, n, n-1, \ldots, \left\lfloor \frac{(j-1)n}{j} \right\rfloor + 1.$$

Permutations in Conjecture 3.2 are examples of so-called layered permutations.

# 4 Circular variant of Roller Coaster permutations

Given $\pi \in S_n$, let $s_i(\pi)$ be the permutation obtained from $\pi$ after cyclically shifting $i$ times to the right. Let us define the following:

$$Y(\pi) = \bigcup_{i=0}^{n-1} \{s_i(\pi)\} \quad \text{and} \quad ct(\pi) = \sum_{\tau \in Y(\pi)} t(\tau).$$

For example, $ct(1324) = t(1324)+t(4132)+t(2413)+t(3241) = 9+10+11+10 = 40$. Let $ct_{max}(n)$ be defined as $\max_{\pi \in S_n} ct(\pi)$. A permutation $\pi \in S_n$ is called a *Circular Roller Coaster permutation* if $ct(\pi) = ct_{max}(n)$. Clearly,

$$ct_{max}(n) \leqslant n \sum_{k=3}^{n} \binom{n}{k} \cdot (k-1).$$

We have the following experimental results for $3 \leqslant n \leqslant 13$:

$$ct_{max} : [5, 40, 168, 592, 1783, 5040, 13106, 33472, 82417, 200536, 471628]$$

Obviously, $ct_{max}(n)/n \leqslant t_{max}(n)$.

# 5 Connections with the partition number of a permutation

Given non-negative integers $r$ and $s$, a permutation $\pi$ has an $(r, s)$-partition if it can be partitioned into $r$ increasing subsequences and $s$ decreasing subsequences. We separate blocks of a partition by '|' and in each block, the relative order of integers is maintained as in $\pi$. For example, 51234 has $(1, 1)$-partitions 51|234, 52|134, 53|124, 54|123, and 5|1234. As in the last case, a single number in a block of partition can be considered as a decreasing (or increasing) subsequence. Define:

$$\begin{aligned} P(r,s) &= \{\pi : \pi \text{ has an } (r,s)\text{-partition}\}, \\ p(\pi) &= \min\{m : m = r+s \text{ and } \pi \in P(r,s)\}, \\ p_{max}(n) &= \max_{\pi \in S_n} p(\pi), \text{ and} \\ PS(n) &= \{\tau : p(\tau) = p_{max}(n)\}. \end{aligned}$$

Here, $p(\pi)$ is called the *partition number* of the permutation $\pi$. For example, $\tau = 2143 \notin P(1,1) \cup P(0,1) \cup P(1,0)$, but is in $P(0,2) \cap P(2,0)$, and so $p(\tau) = 2$. Wagner [6] proved that given a permutation $\pi \in S_n$, the decision problem, 'can $\pi$ be partitioned into $m$ monotone subsequences?', is NP-Complete.

Let $\pi \in S_n$ be called an *extension* of $\tau \in S_m$ (where $m \leqslant n$) if $\tau_j = \pi_{t+j}$ for some $t$ with $0 \leqslant t \leqslant n-m$.

**Conjecture 5.1.** For $\pi \in S_n$, $p(\pi) \leqslant \lceil n/2 \rceil$.

**Observation 5.1.** Here we present some computed values of $p_{max}(n)$ for $3 \leqslant n \leqslant 10$:

$$p_{max} : [2, 2, 2, 3, 3, 3, 3, 4].$$

| $n$ | Example permutation | $PS(n) \cap RC(n) \neq \emptyset$ |
|---|---|---|
| 3 | 132 | $Yes$ |
| 4 | 2143 | $Yes$ |
| 5 | 24153 | $Yes$ |
| 6 | 326145 | $No$ |
| 7 | 3517264 | $Yes$ |
| 8 | 43718265 | $Yes$ |
| 9 | 471639285 | $Yes$ |
| 10 | $\{5, 3, 9, 1, 7, 4, 10, 2, 6, 8\}$ | $No$ |

**Question 5.1.** For which $n$, $RC(n) \cap PS(n)$ is non-empty?

# 6 Connections with forbidden subpermutations

Given $\pi \in S_n$ and $\sigma \in S_m$ with $m \leqslant n$, we say that $\pi$ contains the *subpermutation* $\sigma$ if there exists $\phi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ such that $\pi(\phi(i)) < \pi(\phi(j))$ if and only if $\sigma(i) < \sigma(j)$. For example, 532687941 contains 2143 because of its subsequence 5387. If $\pi$ does not contain $\tau$, then we say, $\pi$ *avoids* $\tau$ (see Kitaev [3] for a comprehensive source of results obtained so far on pattern-avoiding permutations). Define:

$$S_n(\sigma) = \{\pi \in S_n : \pi \text{ avoids } \sigma\},$$
$$S(\sigma) = \bigcup_{n=1}^{\infty} S_n(\sigma).$$

Since 2143 has no $(1, 1)$-partition, any permutation that contains 2143, for example 532687941, has no $(1, 1)$-partition. Let $F(r, s)$ be defined as follows:

$$F(r, s) = \text{minimal } \{\sigma : \pi \in P(r, s) \text{ if and only if } \pi \in S(\sigma)\}.$$

Elements of $F(r, s)$ are called *forbidden permutations* with respect to $r$ and $s$. Stankova [5] observed that $F(1, 1)$ is precisely the set $\{2143, 3412\}$. Kézdy et al. [2] showed that $F(r, s)$ is always finite.

**Question 6.1.** For which $r$ does there exist an $n$ such that $RC(n) \cap F(r, r)$ is non-empty?

# 7 A theoretical question

We observe from the exact and conjectured values of $t_{max}(n)$ that the values of the ratio $t_{max}(n + 1)/t_{max}(n)$ for $4 \leqslant n \leqslant 23$, in order, are:

$$[3.364, 2.865, 2.547, 2.419, 2.332, 2.285, 2.232, 2.204, 2.184, 2.175, \\ 2.151, 2.140, 2.133, 2.129, 2.115, 2.110, 2.105, 2.104, 2.093, 2.089]$$

**Question 7.1.** Does the limit

$$\lim_{n\to\infty} \frac{t_{max}(n+1)}{t_{max}(n)}$$

exist and does it equal to 2?

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# A $RC(n)$ for $10 \leqslant n \leqslant 13$

RC(10) = {(5, 3, 9, 1, 7, 4, 10, 2, 8, 6), (5, 8, 2, 10, 4, 7, 1, 9, 3, 6),
(6, 3, 9, 1, 7, 4, 10, 2, 8, 5), (6, 8, 2, 10, 4, 7, 1, 9, 3, 5)};

RC(11) = {(5, 8, 2, 10, 1, 7, 4, 11, 3, 9, 6), (5, 8, 2, 10, 4, 7, 1, 11, 3, 9, 6),
(5, 8, 2, 11, 1, 7, 4, 10, 3, 9, 6), (5, 8, 2, 11, 4, 7, 1, 10, 3, 9, 6),
(5, 8, 3, 10, 1, 7, 4, 11, 2, 9, 6), (5, 8, 3, 10, 4, 7, 1, 11, 2, 9, 6),
(5, 8, 3, 11, 1, 7, 4, 10, 2, 9, 6), (5, 8, 3, 11, 4, 7, 1, 10, 2, 9, 6),
(5, 9, 2, 10, 1, 7, 4, 11, 3, 8, 6), (5, 9, 2, 10, 4, 7, 1, 11, 3, 8, 6),
(5, 9, 2, 11, 1, 7, 4, 10, 3, 8, 6), (5, 9, 2, 11, 4, 7, 1, 10, 3, 8, 6),
(5, 9, 3, 10, 1, 7, 4, 11, 2, 8, 6), (5, 9, 3, 10, 4, 7, 1, 11, 2, 8, 6),
(5, 9, 3, 11, 1, 7, 4, 10, 2, 8, 6), (5, 9, 3, 11, 4, 7, 1, 10, 2, 8, 6),
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(6, 3, 10, 1, 8, 5, 11, 2, 9, 4, 7), (6, 3, 10, 1, 11, 5, 8, 2, 9, 4, 7),
(6, 3, 10, 2, 8, 5, 11, 1, 9, 4, 7), (6, 3, 10, 2, 11, 5, 8, 1, 9, 4, 7),
(6, 4, 9, 1, 8, 5, 11, 2, 10, 3, 7), (6, 4, 9, 1, 11, 5, 8, 2, 10, 3, 7),
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(6, 4, 10, 2, 8, 5, 11, 1, 9, 3, 7), (6, 4, 10, 2, 11, 5, 8, 1, 9, 3, 7),
(6, 8, 2, 10, 1, 7, 4, 11, 3, 9, 5), (6, 8, 2, 10, 4, 7, 1, 11, 3, 9, 5),
(6, 8, 2, 11, 1, 7, 4, 10, 3, 9, 5), (6, 8, 2, 11, 4, 7, 1, 10, 3, 9, 5),
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(6, 9, 2, 10, 1, 7, 4, 11, 3, 8, 5), (6, 9, 2, 10, 4, 7, 1, 11, 3, 8, 5),
(6, 9, 2, 11, 1, 7, 4, 10, 3, 8, 5), (6, 9, 2, 11, 4, 7, 1, 10, 3, 8, 5),
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(6, 9, 3, 11, 1, 7, 4, 10, 2, 8, 5), (6, 9, 3, 11, 4, 7, 1, 10, 2, 8, 5),
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(7, 4, 10, 2, 8, 5, 11, 1, 9, 3, 6), (7, 4, 10, 2, 11, 5, 8, 1, 9, 3, 6)};

RC(12) = {(6, 4, 10, 2, 12, 5, 8, 1, 11, 3, 9, 7), (6, 9, 3, 11, 1, 8, 5, 12, 2, 10, 4, 7),
(7, 4, 10, 2, 12, 5, 8, 1, 11, 3, 9, 6), (7, 9, 3, 11, 1, 8, 5, 12, 2, 10, 4, 6)};

RC(13) = {(6, 10, 2, 9, 4, 13, 1, 8, 5, 12, 3, 11, 7), (6, 10, 2, 9, 5, 13, 1, 8, 4, 12, 3, 11, 7),
(6, 10, 2, 12, 4, 8, 1, 13, 5, 9, 3, 11, 7), (6, 10, 2, 12, 5, 8, 1, 13, 4, 9, 3, 11, 7),
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(6, 10, 3, 12, 4, 8, 1, 13, 5, 9, 2, 11, 7), (6, 10, 3, 12, 5, 8, 1, 13, 4, 9, 2, 11, 7),
(6, 11, 2, 9, 4, 13, 1, 8, 5, 12, 3, 10, 7), (6, 11, 2, 9, 5, 13, 1, 8, 4, 12, 3, 10, 7),
(6, 11, 2, 12, 4, 8, 1, 13, 5, 9, 3, 10, 7), (6, 11, 2, 12, 5, 8, 1, 13, 4, 9, 3, 10, 7),
(6, 11, 3, 9, 4, 13, 1, 8, 5, 12, 2, 10, 7), (6, 11, 3, 9, 5, 13, 1, 8, 4, 12, 2, 10, 7),
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(7, 4, 12, 2, 9, 6, 13, 1, 10, 5, 11, 3, 8), (7, 4, 12, 2, 10, 6, 13, 1, 9, 5, 11, 3, 8),

$$
\begin{aligned}
&(7,4,12,5,9,1,13,6,10,2,11,3,8),(7,4,12,5,10,1,13,6,9,2,11,3,8),\\
&(7,10,2,9,4,13,1,8,5,12,3,11,6),(7,10,2,9,5,13,1,8,4,12,3,11,6),\\
&(7,10,2,12,4,8,1,13,5,9,3,11,6),(7,10,2,12,5,8,1,13,4,9,3,11,6),\\
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&(8,4,12,2,9,6,13,1,10,5,11,3,7),(8,4,12,2,10,6,13,1,9,5,11,3,7),\\
&(8,4,12,5,9,1,13,6,10,2,11,3,7),(8,4,12,5,10,1,13,6,9,2,11,3,7)\}.
\end{aligned}
$$